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**Manuscript title:** Discussion: Undrained cylindrical cavity expansion in anisotropic critical state soils

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**Contribution by Jorge Castro and Nallathamby Sivasithamparam**

The discussers congratulate the authors for their solution and the footnote citing the recent discussers' solution (Sivasithamparam & Castro 2018), which clarifies that the discussers' solution was published (strictly speaking, made available online) during the revision of the authors' manuscript. The discussers would like to briefly comment on the differences between the two solutions (rotational hardening law) and the way the results are presented (undrained shear strength and yield locus).

**Rotational hardening law**

Both solutions extend the isotropic solution by Chen & Abousleiman (2012) considering an anisotropic critical state model with the same rotated ellipse as the yield surface (Dafalias 1986). The main difference between the two solutions is the rotational hardening law that they consider. Sivasithamparam & Castro (2018) use the rotational hardening law proposed by Wheeler et al. (2003), i.e. S-CLAY1 model, while the authors use the original one proposal by Dafalias (1986, 1987). Dafalias & Taiebat (2013) present a detailed analysis of four rotational hardening laws, including the two above. Here, just two differences for this particular case are mentioned:

(1) The original rotational hardening law by Dafalias (1986, 1987) does not predict a unique critical state line (CSL). Dafalias & Taiebat (2013) suggest a minor modification in the rotational hardening law formulation to achieve uniqueness of the CSL. Consequently, for the studied cavity expansion problem, the authors' solution does not predict a unique inclination of the yield surface at CS (e.g. Fig. 11 of the authors' paper), nor the stress state (e.g. Fig. 10), while the discussers' solution predicts a unique inclination and stress state at CS that can be analytically obtained (Sivasithamparam & Castro 2018).

(2) Sivasithamparam & Castro (2018) show how the yield surface rotates from the initial "vertical" anisotropy (axis of the yield surface in the triaxial compression plane, which corresponds to a Lode angle of $\theta$=7π/6 using the authors' definition of the Lode angle, Fig. 5) towards a "radial" anisotropy (axis of yield surface in the plane strain plane, $\theta$=10π/6). In the authors' case, minor fabric changes are predicted (Fig. 11) and rotation of the yield surface does not reach $\theta$=10π/6. Thus, the effective vertical stress is not the intermediate stress at critical state ($\sigma'_z \neq \sigma'_r + \sigma'_\theta$), as in Li et al. (2016).

**Undrained shear strength**

A difference between the presentation of results in the two papers is the way the undrained shear strength ($s_u$) is defined to normalise the results (e.g. Figs. 7 and 8). The authors use the "isotropic" value of $s_u$. and consequently, the normalized values of the initial stresses for isotropic and anisotropic cases are the same in Figs. 7 and 8. The "anisotropic" value of $s_u$ depends on the initial inclination of the yield surface. For example, Sivasithamparam & Castro (2018) present the analytical equation to get $s_u$ for the S-CLAY1 model, and use the corresponding $s_u$ values for each initial inclination of the yield surface ($\alpha_0$). In the authors' case, this is not possible due to the lack of uniqueness of the CSL introduced by the rotation hardening law (point 1 above).

**Yield locus**

An isotropic yield surface may be plotted using $p'$ and $q$ stress invariants without loss of generality. However, for an anisotropic yield surface, it implies assuming a specific value of the Lode angle. As explained by the authors using Fig. 4, there is a continuous variation of the Lode angle during cavity expansion. So, termination of the elastic deformation is not

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located at the initial anisotropic yield surface corresponding to $\theta$=π/6 and 7π/6, but rather at some other elliptical cut taken through the initial yield surface yet with a changed value of $\theta$. That is why Sivasithamparam & Castro (2018) decided to introduce a new stress invariant $\bar{q}$ and plotted yield loci in the deviatoric plane (π-plane). Yield loci in the $p'$-$q$ plane were plotted for a constant Lode angle, namely $\theta$=7π/6. In contrast, the authors plot yield loci in the $p'$-$q$ plane for different Lode angles. For example, initial yield locus in Fig. 12b corresponds to Lode angle of point C (θ≈9π/6).

**Authors' reply**

The authors would like to thank Drs. Castro and Sivasithamparam for their interest in the paper. They are also to be congratulated on presenting a similar undrained solution for the cavity expansion problem in anisotropic critical state soils (Sivasithamparam & Castro, 2018), which was published online in *Acta Geotechnica* during the revision of this paper (Chen & Liu, 2018) [more accurately in the second round of review process, after making the first revision submission to *Geotechnique*]. The discussers comment on the differences between their solution and the one derived by the authors from the following three aspects: the rotational hardening law involved, and the way the results being presented in terms of the undrained shear strength and yield locus, respectively. These points raised are addressed as follows.

**<u>Rotational hardening law</u>**

It is recognized that both the two solutions by the authors (Chen & Liu, 2018) and the discussers (Sivasithamparam & Castro, 2018) are the extension of Chen & Abousleiman's (2012) work from isotropic to anisotropic modified Cam Clay with the use of a same form of ellipsoidal yield surface, and that the main difference is pertinent to/resulting from the rotational hardening rules considered in the respective anisotropic models (Dafalias, 1987; Wheeler et al., 2003) used for the development of analytical solutions. Our purpose here is to propose a rigorous analytical solution for the cavity expansion problem in generic anisotropic critical state soils, instead of the constitutive model itself. Hence, we have restricted attention to Dafalias' (1987) model.

We agree that our solution by using Dafalias' (1987) original rotational hardening law, which achieve multiple critical state lines (CSLs) in the void ratio versus mean effective stress plane as elaborated in Dafalias & Taiebat (2013), cannot predict a unique orientation of the yield surface at the critical state. This is in contrast with the discussers' solution where a unique critical state inclination of the yield surface has been observed for different values of $\mathrm{OCR}$ considered. However, the authors do not agree with the discussers' statement regarding the uniqueness of their obtained stresses $\sigma'_{r,f}$, $\sigma'_{\theta,f}$, and $\sigma'_{z,f}$ as well at the failure critical state. Note that these stress components, as clearly shown in Fig. 6 of Sivasithamparam & Castro (2018), are only acquiring unique (constant) values when normalised with respect to the undrained shear strength $s_u$, while not their absolute magnitudes as $s_u$ indeed varies with the $\mathrm{OCR}$ value itself (see Table 2 in the discussers' paper).

With regard to the vertical effective stress, it is true that in the discussers' solution, $\sigma'_{z,f}$ at the critical state will become equal to the average of the radial and tangential stresses, i.e., $\sigma'_{z,f} = \frac{\sigma'_{r,f}+\sigma'_{\theta,f}}{2}$. Nevertheless, it is important to point out that such a simple and plausible relationship in fact is not a direct consequence of the plane strain conditions, as claimed by the discussers (Sivasithamparam & Castro, 2018). Nor should it be interpreted as equivalent to the intermediate stress condition imposed to the vertical stress $\sigma'_{z,f}$, the latter of which only requires $\sigma'_{\theta,f} < \sigma'_{z,f} < \sigma'_{r,f}$ (or $\sigma'_{r,f} < \sigma'_{z,f} < \sigma'_{\theta,f}$). The same can also be said to the three critical state values of the rotational hardening parameters $\alpha_{r,f}$, $\alpha_{\theta,f}$, and $\alpha_{z,f}$, for which the discussers have asserted that $\alpha_{z,f} = \frac{\alpha_{r,f}+\alpha_{\theta,f}}{2}$ holds as a result again of the plain strain conditions. The authors cannot find the logic behind these more or less taken-for-granted relationships/statements, they however may be rigorously proved as follows.

With some minor algebraic manipulations, the three equations of (41) in Sivasithamparam & Castro (2018) can be reformed as

$$\frac{D}{Dr}(\sigma_r' + \sigma_\theta' - 2\sigma_z') = -\frac{(1-2\nu)(n_r - n_\theta)(n_r + n_\theta - 2n_z)\mathcal{H}}{E\Gamma r} \tag{57}$$

with

$$n_r = \frac{\partial f_y}{\partial \sigma_r'} = \frac{p'(M^2-\alpha^2-\bar{\eta}^2)}{3} + \{3[\sigma_r' - (\alpha_r^d + 1)p'] - s_r\alpha_r^d - s_\theta\alpha_\theta^d - s_z\alpha_z^d\} \tag{58}$$

$$n_\theta = \frac{\partial f_y}{\partial \sigma_\theta'} = \frac{p'(M^2-\alpha^2-\bar{\eta}^2)}{3} + \{3[\sigma_\theta' - (\alpha_\theta^d + 1)p'] - s_r\alpha_r^d - s_\theta\alpha_\theta^d - s_z\alpha_z^d\} \tag{59}$$

$$n_z = \frac{\partial f_y}{\partial \sigma_z'} = \frac{p'(M^2-\alpha^2-\bar{\eta}^2)}{3} + \{3[\sigma_z' - (\alpha_z^d + 1)p'] - s_r\alpha_r^d - s_\theta\alpha_\theta^d - s_z\alpha_z^d\} \tag{60}$$

where *D* denotes the material derivative; and all the other notations in the above equations have exactly the same definitions as those appearing in Sivasithamparam & Castro (2018). Substituting Eqs. (58)–(60) into Eq. (57), one obtains

$$\frac{D}{Dr}(\sigma_r' + \sigma_\theta' - 2\sigma_z') = -\frac{3(1-2\nu)(n_r - n_\theta)\mathcal{H}}{E\Gamma r}\{(\sigma_r' + \sigma_\theta' - 2\sigma_z') - p'(\alpha_r^d + \alpha_\theta^d - 2\alpha_z^d)\} \tag{61}$$

At the critical state, the deviatoric rotational hardening parameters according to Eq. (33) in Sivasithamparam & Castro (2018) can be determined as

$$\alpha_{r,f}^d = \frac{\sigma_{r,f}'}{3p_f'} - \frac{1}{3}, \quad \alpha_{\theta,f}^d = \frac{\sigma_{\theta,f}'}{3p_f'} - \frac{1}{3}, \quad \alpha_{z,f}^d = \frac{\sigma_{z,f}'}{3p_f'} - \frac{1}{3} \tag{62}$$

Upon the substitution of Eq. (62), Eq. (61) pertinent to the critical state conditions may be expressed as

$$\frac{D}{Dr}(\sigma_r' + \sigma_\theta' - 2\sigma_z') = -\frac{3(1-2\nu)(n_r - n_\theta)\mathcal{H}}{E\Gamma r}\left\{\frac{2}{3}(\sigma_r' + \sigma_\theta' - 2\sigma_z')\right\} \tag{63}$$

which indicates that, to guarantee the constant stress components at the critical state, i.e., $\frac{D}{Dr}(\sigma_r' + \sigma_\theta' - 2\sigma_z') \equiv 0$, the only possible zero term on the right side, $\sigma_r' + \sigma_\theta' - 2\sigma_z'$, must also vanish (Chen & Abousleiman, 2012). The desired relationship $\sigma_{z,f}' = \frac{\sigma_{r,f}' + \sigma_{\theta,f}'}{2}$ is therefore justified. Furthermore, in light of Eq. (62), it can be easily seen that $\alpha_{z,f}^d$ must also be the mean of the other two deviatoric anisotropic variables $\alpha_{r,f}^d$ and $\alpha_{\theta,f}^d$, i.e., $\alpha_{z,f}^d = \frac{\alpha_{r,f}^d + \alpha_{\theta,f}^d}{2}$. Hence, evidently $\alpha_{z,f} = \frac{\alpha_{r,f} + \alpha_{\theta,f}}{2}$.

It should be remarked that in Chen & Liu (2018), no such simple relationships as Eq. (62) will exist due to the lack of uniqueness/determinacy of $\alpha_{r,f}^d$, $\alpha_{\theta,f}^d$, and $\alpha_{z,f}^d$ at the critical state. That well explains why $\sigma_{z,f}' \neq \frac{\sigma_{r,f}' + \sigma_{\theta,f}'}{2}$ and $\alpha_{z,f} \neq \frac{\alpha_{r,f} + \alpha_{\theta,f}}{2}$ in the authors' solution. Nevertheless, it is interesting to note that the above equation (61) actually still holds true in our solution while using Dafalias' (1987) rotational hardening rule. We thus get the following result connecting the critical stresses with the associated anisotropic hardening variables

$$\frac{\sigma_{r,f}' + \sigma_{\theta,f}' - 2\sigma_{z,f}'}{p_f'} = \alpha_{r,f} + \alpha_{\theta,f} - 2\alpha_{z,f} \tag{64}$$

**Undrained shear strength**

The primary purpose of Figs. 7 and 8 in Chen & Liu (2018) is to illustrate the influences of the $K_0$ consolidation anisotropy on the stress distributions around the cavity and on the typical stress paths followed in the $p' - q$ plane, for two identical normally consolidated soils. In our opinion, choosing the same "isotropic" value of $s_u$ for both isotropic and anisotropic cases to normalise the resulting stresses in these two figures would make more sense than introducing an additional "anisotropic" value of $s_u$ for the anisotropic solution as advocated by the discussers. This is because the use of the "anisotropic" undrained shear strength for the normalisation, which differs from the "isotropic" one corresponding to the isotropic Cam Clay model, will somewhat mask the true influences of the $K_0$ consolidation anisotropy on the calculated stress responses due to the inconsistency of the normalisation denominator involved.

It seems that in the discussers' solution the "anisotropic" value of the undrained shear strength (plane strain conditions) used for the normalisation are calculated from $s_{u,PS} = \frac{\sigma'_{r,f} - \sigma'_{\theta,f}}{2}$, with $\sigma'_{r,f}$ and $\sigma'_{\theta,f}$ at the critical state being numerically determined following the solution procedure proposed in their paper. If that is the case, then it is difficult to see why the authors' solution cannot be normalised in a similar way in terms of the evaluated critical state stress components as in the discussers' case, although the authors do agree with the non-uniqueness statement of the CSL in association with the currently involved rotational hardening law (Dafalias, 1987).

**Yield locus**

The authors fully agree with the discussers' suggestion that, to more clearly and effectively demonstrate the effective stress path for the cavity expansion in an anisotropic soil, it is favorable to present the projections of the stress path on both the $p' - q$ plane and the deviatoric ($\pi$) plane. As a matter of fact, in a more recent paper on the drained cavity analysis (Liu & Chen, 2018), we have already done so to give a clearer insight into the stress path evolution during the cavity expansion process. However, for the plotting of effective stress path in the $p' - q$ plane, our preference is to include the yield locus passing through the initial yielding stress point $C$, i.e., the particular intersection curve between the initial ellipsoidal yield surface and a plane containing the hydrostatic axis and point $C$, see Figs. 12(b) and 12(c) in Chen & Liu (2018). Such a deliberate treatment in presenting the initial

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yield locus should possess advantage over the elliptical cut corresponding to a constant Lode angle of $\theta = \frac{7\pi}{6}$ as adopted by the discussers, since according to their Figs. 8(b) and 8(c) it appears that yielding may take place for a stress state even within the initial yield surface which is sort of misleading.

**References**

Chen, S. L. & Abousleiman, Y. N. (2012). Exact undrained elasto-plastic solution for cylindrical cavity expansion in modified Cam Clay soil. *Geotechnique* **62**, No. 5, 447-456.

Chen, S. L. & Liu, K. (2018). Undrained cylindrical cavity expansion in anisotropic critical state soils. *Geotechnique*, ahead of print. doi: 10.1680/jgeot.16.p.335.

Dafalias, Y.F. (1986). An anisotropic critical state soil plasticity model. *Mech. Res. Commun.* 13, 341–47.

Dafalias, Y. F. (1987). An anisotropic critical state clay plasticity model. *Proceedings of the Constitutive Laws for Engineering Materials: Theory and Applications (eds C. S. Desai, E. Krempl, P. D. Kiousis and T. Kundu)*, Tucson, pp. 513-521. New York, NY, USA: Elsevier.

Dafalias, Y. F. & Taiebat, M. (2013). Anatomy of rotational hardening in clay plasticity. *Geotechnique* **63**, No. 16, 1406-1418.

Li, L., Li, J. & Sun, D. (2016). Anisotropically elasto-plastic solution to undrained cylindrical cavity expansion in K0-consolidated clay. *Comput. Geotech.* 73: 83-90.

Liu, K. & Chen, S. L. (2018). Analysis of cylindrical cavity expansion in anisotropic critical state soils under drained conditions. *Can. Geotech. J.*, published online. doi: 10.1139/cgj-2018-0025.

Sivasithamparam, N. & Castro, J. (2018). Undrained expansion of a cylindrical cavity in clays with fabric anisotropy: theoretical solution. *Acta Geotech*. **13**, No. 3, 729-746.

Wheeler, S. J., Naatanen, A., Karstunen, M. & Lojander, M. (2003). An anisotropic elastoplastic model for soft clays. *Can. Geotech. J.* **40,** No. 2, 403-418.