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# NUMERICAL WAVENUMBER INTEGRATION FOR 2.5D WAVE EQUATION SOLUTION

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**Abstract.** *In this study, we propose and implement a convergence scheme for the numerical wavenumber integration in the context of 2.5D FE solution and validate it through a numerical example. In brief, the scheme is to estimate the convergence based on the relative difference between two interpolations for a given set of wavenumbers, which is done iteratively by refining the wavenumber sampling until the relative difference becomes no greater than a specified value. We evaluate the performance and convergence of the numerical integration scheme by comparing the results of the responses of a tunnel in a layered ground due to a concentrated harmonic load using both a full 3D model and a 2.5D model, including computational time aspect. Finally, we consider applying the same scheme to other relevant numerical integrations.*


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## 1 INTRODUCTION

Most problems in structural and soil dynamics are defined in 3D space. However, in a mathematical framework quite a few among those problems can be simplified in such a way that the geometry and material properties (i.e. density, stiffness, damping, etc.) of the media of interest are invariant along a particular axis. It is then possible to solve some of these 3D problems via the so-called 2.5D numerical approach (e.g. Galvin et al., 2010). For instance, wave propagation due to moving load along a tunnel structure built in layered soil media is a good example where the given geometry and material properties are often assumed constant along the track direction (Say $y$-axis), while they vary along the other two directions ($x$ and $z$). Thanks to the $y$-direction invariance of both geometry and property, all the differentiations with respect to the $y$ coordinate in the 3D wave equation can be replaced by the $y$-direction wavenumber ($k_y$) without any loss of accuracy in describing the features in the 3D problems. Then, as in discrete methods such as finite element (FE) method, the solution can be obtained by a spatial discretization only along the $x$- and $z$-axes, effectively reducing the number of degrees of freedom (DOF) to solve. On the other hand, this formulation requires that the resulting 2D FE equation be solved for many wavenumbers $k_y$ in the FE context. This can be time-consuming but is still much less demanding than a full 3D FE analysis, provided that one employs a procedure that ensures efficient convergence of the associated wavenumber integration. Therefore, it is important to optimize the sampling and number of $k_y$'s to solve so that one can calculate accurate and converged results, while the computational time is reasonable. Particularly for calculation of high frequency (~100Hz) dynamic response at far offset along the $y$-axis, the 2.5D FE solution indeed becomes essential in comparison to a full 3D FE solution due to the computational memory requirement.

In this study, we implement a convergence scheme for the numerical wavenumber integration in the context of 2.5D FE solution and validate it through a numerical example. In brief, the scheme is to estimate the convergence based on the relative difference between two interpolations (e.g. linear and spline) for a given set of wavenumbers, which is done iteratively by refining the wavenumber sampling until the relative difference becomes no greater than a specified value (e.g. 0.01). We discuss the performance and convergence of the numerical integration scheme by considering various aspects. At the end, we also discuss to apply the same scheme to other relevant numerical integrations e.g. wavenumber integration for the fully analytical solution.

All the FE calculations in the current study are performed by means of the commercial software *COMSOL Multiphysics*™. The relevant 2.5D wave equation is implemented and solved in the module of *2D Coefficient Form PDE*. In order to avoid artificial reflection from the computational domain boundary, a perfectly matched layer technique (Park and Kaynia, 2017) is deployed in the current study.

## 2 PROPOSED WAVENUMBER INTEGRATION METHOD

The numerical wavenumber integration is a common and necessary step in calculation of the analytical-solution for elastic wave propagation in layered media (e.g. Apsel and Luco, 1983; Park and Kaynia, 2018) to obtain dynamic response in the spatial domain from the wavenumber domain. There are two associated challenges: 1) handling of the pole singularities in the kernels of wavenumber integration; and 2) controlling of the convergence of numerical integration. The former challenge, which is the most critical in the analytical solution, is due to the extremely large and singular value of kernels at the propagation poles. A similar procedure for numerical wavenumber integration is required in the 2.5D FE solution. However, the challenge is related mainly to the convergence control, because the pole singularity doesn't exist or is not so strong in the 2.5D FE solution. In the following, we discuss these features through an example of wave response in a homogeneous half-space.

Figure 1 compares two kernels calculated via a fully analytical solution (red thin solid line) by Park and Kaynia (2018) and the corresponding 2.5D FE solution (blue thick solid line). Both of the kernels are for the same homogeneous half-space subjected to a harmonic vertical load with frequency 100 Hz applied at the surface. The kernels are calculated exactly at the loading point. It is clearly shown in Figure 1 that the kernel of the 2.5D FE solution looks rather moderate without any strong singularity near the Rayleigh pole (black thin dashed vertical line). On the other hand, that of the analytical solution shows a strong and sharp singularity near the Rayleigh pole, subsequently requiring very-densely-sampled wavenumbers to calculate accurately. It is shown that the challenge in relation to the pole singularity is less significant (or almost negligible) in the 2.5D FE solution than the analytical solution. Therefore, it is expected that interpolation of the kernels of the 2.5D FE solution is feasible, even near the poles, which in turn can save significantly the computation effort in the 2.5D FE solution. Yet, we still need to make sure the convergence of the wavenumber integration by optimizing the sampling and number of wavenumbers to include into the 2.5D FE calculation.

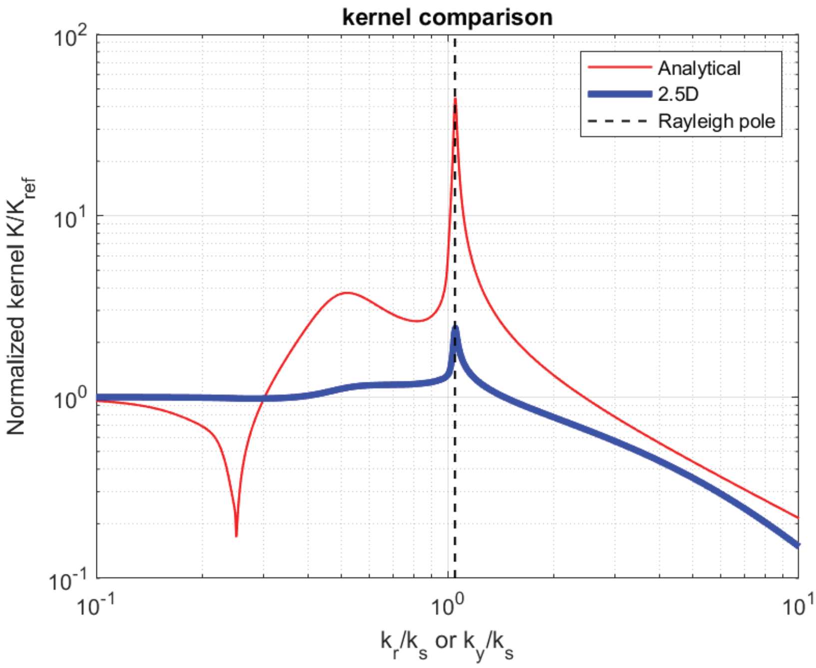



Figure 1. Comparison of two kernels calculated by means of analytical and 2.5D solutions, normalized with respect to the reference kernel calculated at the zero wavenumber. Note that the *x*-axis is the normalized wavenumber with respect to the shear wavenumber $k_s$; and $k_r$ and $k_y$ are the radial- and y-direction wavenumbers, respectively, used in the analytical and 2.5D FE solutions.

Based on the observations made through the example in Figure 1, we propose a convergence controlling scheme. The scheme a simple approach, consisting of the following steps:

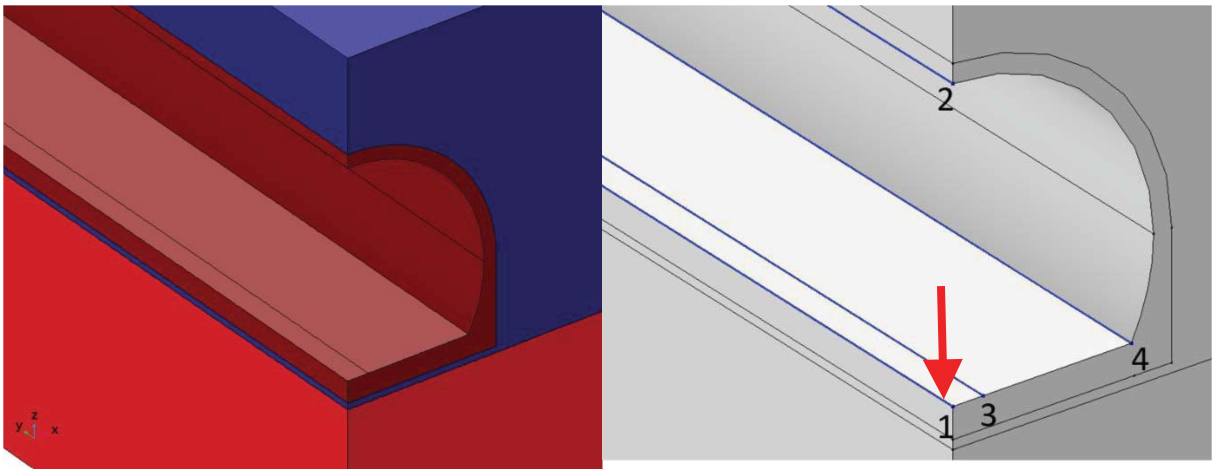
1. Initialize a wavenumber sampling of $k$=[0:$\Delta k$:$k_{max}$]. Here, $k_{max}$ should be so large that the kernels at $k_{max}$ is small enough. Note that as shown in Figure 1, the kernels decay quickly for $k$>$k_{Rayleigh}$. In addition, $\Delta k$ could be large enough such that the total number of $k$'s is no so high e.g. < 10. Alternatively, the sampling can be done logarithmically, while using the same $k_{max}$.
2. Calculate the kernels for the given wavenumber sampling $k$ by 2.5D FE solution procedure.

3. Interpolate the kernels by means of two different interpolations (e.g. linear and spline).
4. Estimate the relative differences between the interpolated values of the two different interpolation methods at the middle points of individual wavenumber sampling intervals.
5. Collect the middle-point wavenumbers (say $k_{>0.01}$) where the relative differences are greater than a specified criterion (e.g. 0.01).
6. Re-calculate the true kernels for those collected middle-point wavenumbers $k_{>0.01}$ by 2.5D FE solution procedure.
7. Repeat Steps 3-6 by updating $k$ with accumulatively adding $k_{>0.01}$, until $k_{>0.01}$ does not exist.

Once Step 7 above is completed, we perform an inverse Fourier transform from $k_y$-domain to $y$-domain in order to calculate the final wave response in the spatial domain. We have found through numerical experiments that the number of wavenumbers after Step 7 is around a few 100's. If, on the other hand, we in a brute force manner use an equally-spaced wavenumber sampling for k=[0,$k_{max}$], the total number of wavenumbers could be in the order of $>10^4$.

## 3 NUMERICAL EXAMPLES

We validate the wavenumber integration scheme proposed in the study by solving a simple example of a tunnel embedded in a two-layered soil medium. Figure 2(a) shows the full 3D model described with 2 symmetric planes used at $x$=0m and $y$=0m. The tunnel is located in the upper soil layer of 16m thickness, and the tunnel bottom is close to the lower bedrock layer of infinite thickness. The thickness of the tunnel wall varies between 0.6m to 1.0m along the perimeter, shown in Figure 2. The harmonic excitation load of 100 Hz is applied in the middle of the tunnel base and in the vertical direction, red arrow in Figure 2(b). The material properties and geometry in the example problem is shown in Table 1. In addition, Figure 2(b) shows four lines highlighted in blue, where we extract the displacement results and compare with a full 3D FE simulation. Figure 3(a) and (b) show the FE meshes used in calculation, respectively, for the 2.5D and 3D analyses. The numbers of DOFs used in each model are around $43\times10^3$ and $1250\times10^3$, respectively.

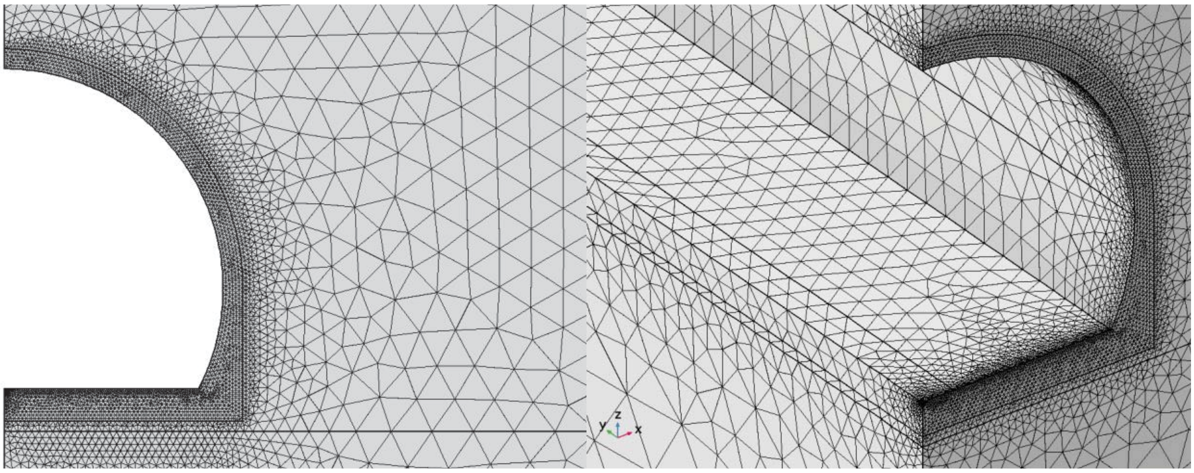



(a) (b)

Figure 2. (a) Example problem of a tunnel imbedded into two-layered soil medium (modelled with two symmetric planes at $x$=0m and $y$=0m); (b) 4 lines (blue) on the tunnel surface where responses are calculated and compared. Red arrow shows where the vertical point load of 100Hz is applied on Line 1.

| Material | Density [kg/m$^3$] | Shear/Compression velocity [m/s] | Thickness [m] |
|---|---|---|---|
| Upper layer | 1500 | 500/2000 | 16 |
| Lower layer | 3000 | 2000/4000 | inf |
| Concrete tunnel | 2300 | 2128/3475 | 0.6 (bottom: 1.0) |

Table 1. Material properties and geometry in the example problem in analysis. A uniform viscous damping of 1% is applied to all the materials.

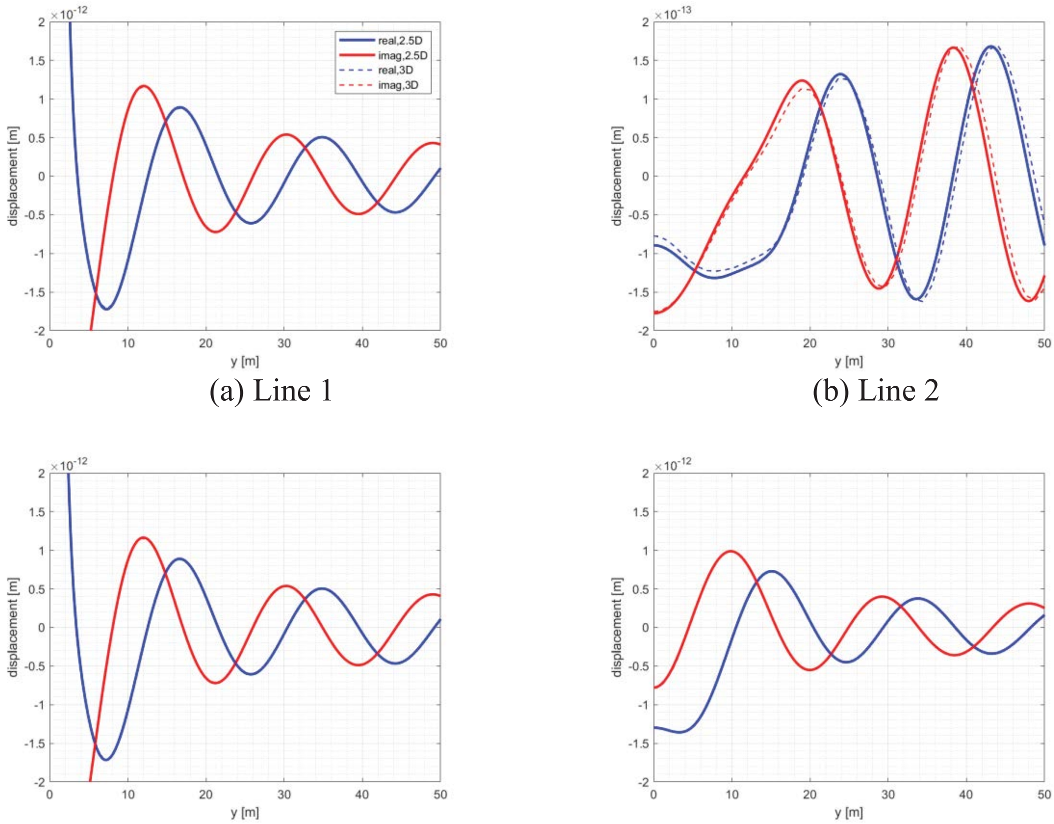


(a) (b)

Figure 3. FE meshes used: (a) for 2.5D simulation; (b) for 3D simulation

Figure 4 compares the two results of the 2.5D and 3D FE simulations, calculated along the 4 lines highlighted with blue lines in Figure 2(b). The solid and dashed lines are, respectively, for the 2.5D and 3D FE solutions, and the blue and red colors denote, respectively, the real and imaginary parts of the time-harmonic response. It is clearly shown that the agreement between the 2.5D and 3D FE simulations is perfect for Line 1, 3 and 4. Namely, the solid and dashed lines are exactly on the top of each other. The agreement for Line 2 (line along the tunnel crown) is also very good but not as much as the other lines, which is believed to have to do with the wavenumber sampling near the Rayleigh (or generally, wave propagation) pole(s). The improvement on this issue is under investigation.

It is mentioned earlier that $k_{max}$ should be so large that the kernels at $k_{max}$ is small enough, making sure that the numerical integration is converged. Furthermore, we have also found out that $k_{max}$ depends on the relative position of (or distance between) load and receiver points. Figure 5 shows the normalized kernels for the four response lines in Figure 2(b) whose distances to the vertical load are different. Line 1 has zero distance to the load point, and Lines 3, 4 and 2 have 1m, 6m and ca 11m distance, respectively. It is easily noticed in Figure 5 that the decaying rate of the kernels is proportional to the distance between the load and response points. Therefore, we may conclude that the longer the distance is, the smaller $k_{max}$ is needed.




(a) Line 1 (b) Line 2

(c) Line 3 (d) Line 4

Figure 4. Comparison of 2.5D (solid lines) and 3D (dashed lines) simulation results in displacement along 4 lines on the tunnel surfaces (defined in Figure 2). Note that the Y-axis scale of Plot (b) is 10 time smaller than that of the other three plots.

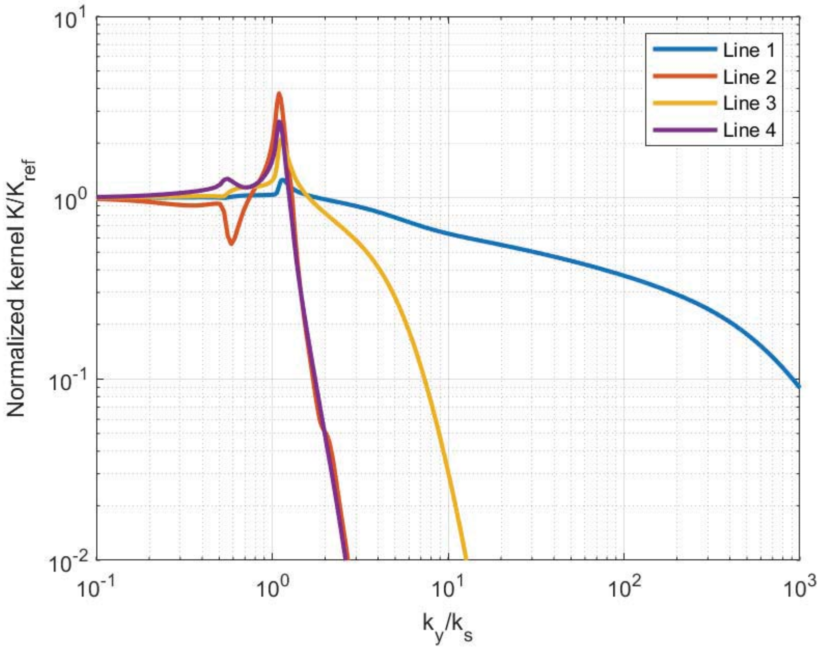



Figure 5. Comparison of kernels calculated for the four lines on the tunnel surfaces (defined in Figure 2). Note that the kernels are normalized with respect to the reference kernel calculated at the zero wavenumber. In addition, the *x*-axis is the normalized wavenumber with respect to the shear wavenumber $k_s$ of concrete (~0.295).

## 4 CONCLUSIONS

In this study, we propose a simple scheme with which we can efficiently integrate the wavenumber integration in the context of 2.5D FE solution for a full 3D wave equation. The scheme consists of 7 steps and controls iteratively the convergence of the integration. The satisfactory performance of the proposed integration scheme is demonstrated by comparing the results of the responses of a tunnel in a layered ground due to a concentrated harmonic load using both a full 3D model and a 2.5D model. In addition to comparison of the displacements, information about computational time is provide. One of the key parameters in the proposed scheme is the maximum wavenumber to be used in the analysis ($k_{max}$). Analyses indicate that the longer the distance between load and response points, the smaller $k_{max}$ is required.

Herein, we have applied successfully the proposed integration technique to the 2.5D FE wave equation. Nevertheless, it is worthwhile to mention that the same technique can also be applied to any numerical integration e.g. numerical wavenumber integration for the fully analytical solution by Park and Kaynia (2018), replacing a brute force integration with equally-spaced wavenumbers and saving computation time by >10 times. A related study is in progress.

## 5 ACKNOWLEDGEMENT

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