FINITE ELEMENT MODELING OF SNOW-PACK LYING ON A SLOPE, CONSIDERING SNOW AS ISOTROPIC COMPRESSIBLE VISCOPLASTIC MEDIA

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ABSTRACT : A simple extension of the von Mises plasticity is proposed in which the equivalent stress is defined as a function of deviatoric and hydrostatic stresses. Non-linearity is accounted by extending Norton-Hoff equation for incompressible material to snow, a porous material. For developing a multi-axial constitutive equation a complementary viscoplastic potential, expressed as a function of the equivalent stress tensor, is introduced. With this potential the strain-rate tensor is obtained. Coefficients of the constitutive equation were computed with the help of experimental data. This constitutive equation is utilized to investigate the stress and velocity distribution in a snow-pack with a weak layer on a uniform slope. This weak layer has a super weak zone, responsible for initiating avalanches. Self-weight of snow is the only external force being considered. The finite-element code, based upon a plane-strain idealization, is used. Linear constitutive equation is used to give an initial guess as Newton-Raphson method has been employed for solving the system of non-linear equations. For non-linear case convergence criterion is implemented for both unknown velocities and residual forces. The effects of super-weak zone, thickness of weak layer and length of snow slab on shear stresses and deformation rates have been studied.

INTRODUCTION

In the progression of analysis of steeply sloping snow-packs, emphasis has been placed upon the state of stress in attempts to understand the mechanisms leading to avalanche release. Hafeli (1963) hypothesizes a material transformation based upon change of axes of principal stress with depth leading to deep slab failures. Perla (1971) develops a proof of the existence of a weak basal plane as a prerequisite for avalanche release, drawing upon the observed characteristics of the crown-region fracture geometry, and the associated stress distribution. Smith and Curtis (1975) demonstrate the applicability of numerical methods to slab analysis, and model stress distribution and failure prediction based upon material layering and local geometric effects. A first attempt at deformation prediction was made by Lang and Brown (1975) in which a nominal visco-elastic material idealization is used in conjunction with an assumed shear weakness in the basal plane to predict stress and strain-rate distribution theoretically. Lang and Sommerfeld (1977) compared analytically predicted snow-pack deformation with actual measured deformations obtained by use of strain gauges mounted in creeping snow-pack.

The basis for the present study is the work done by Bader and Salm (1989), in which they have presented numerical and analytical investigations about stress and strain-rate distribution in a snow-pack on a uniform slope containing an initial crack of infinite length across the slope. They have used a linear constitutive equation for all the three layers. They concluded that a snow slab cannot fracture until a critical strain rate is reached or exceeded. Natural release of snow slab avalanches seem impossible without a-priori existing perturbations in the weak layer. However, this is a necessary but not sufficient condition for avalanche formation.

Chenot et al [1990] developed the constitutive equation for isotropic compressible plasticity. The generalization of the compressible plasticity is to use the Norton-Hoff equation for incompressible material and extend it to porous material. They have given finite element models for hot isostatic pressing and forging of powder metals.

BACKGROUND

To study the effect of a weak layer in a snow-pack on the stresses and deformation rates, a simple three layer model proposed by Bader and Salm (1989) has been considered. The significant difference is that in the present case a nonlinear equation has been developed and utilized. Angle of slope has been increased to 50^o as applicable to the formation zone of an avalanche.

NONLINEAR CONSTITUTIVE EQUATION

It was assumed that creep movement of snow follows viscoplastic behaviour, because snow behaves as a compressible material. Though, Bader and Salm (1989) solved the problem using a linear viscoelastic constitutive law, in reality snow is a nonlinear viscoplastic granular material and a suitable law has to be used. Since metal powders behave in viscoplastic nature, it is felt that a constitutive law used for them can be extended to snow and attempts are being made in this direction.

MATERIAL BEHAVIOUR

A simple extension of the von Mises plasticity (Chenot et al) has been considered in which the equivalent stress is defined by

$\bar{\sigma} = (\frac{3c}{2} s:s + f(tr(\sigma))^2)^{\frac{1}{2}}$ where σ is the stress tensor, s is the deviatoric stress tensor

$tr(\sigma) = \sum_i \sigma_{ii}$; $s:s = \sum_{ij} s_{ij} s_{ij}$

c and f are the functions of the relative density ρ_r of the material. The definitions of equivalent stress and equivalent strain reduce to that for incompressible plasticity for c(1) = 1 and f(1) =0. The equivalent strain rate is defined by

$\dot{\bar{\varepsilon}} = \frac{1}{\rho_r}(\frac{2}{3c}\dot{e}:\dot{e} + \frac{1}{9f}(tr(\dot{\varepsilon}))^2)^{\frac{1}{2}}$ where $\dot{e}$ is the deviatoric strain rate tensor

VISCOPLASTICITY

The complementry viscoplastic potential for three dimension (Chenot et al) is defined by

$$\varphi(\dot{\varepsilon}) = \frac{k}{m+1}(\sqrt{3}\,\rho_r \dot{\bar{\varepsilon}})^{m+1}$$

with this potential the stress tensor is immediately obtained by

$$8\sigma = \frac{\partial \varphi}{\partial \dot{\varepsilon}} = \frac{1}{\rho_r \dot{\bar{\varepsilon}}} \frac{\partial \varphi(\rho_r \dot{\bar{\varepsilon}})}{\partial(\rho_r \dot{\bar{\varepsilon}})}[\frac{2}{3c}\dot{e} + \frac{1}{9f}(tr(\dot{\varepsilon}))I]$$

$$= 3K(\sqrt{3}\,\rho_r \dot{\bar{\varepsilon}})^{m-1}[\frac{2}{3c}\dot{e} + \frac{1}{9f}(tr(\dot{\varepsilon}))I]$$

FINITE ELEMENT MODELING

Flow approach has been used since the problem is concerned with large deformation and the elastic deformation is almost negligible (Kobayashi et al). Thickness of the weak layer has been taken to be 0.002 m while top and bottom layers are 1 m thick. In plane strain idealization, 40 m length of the slope has been considered. Strong layers has been divided into 4 elements lengthwise and 3 elements heightwise i.e. 12 elements for top layer and 12 elements for bottom layer. A 2-D 4-noded isoparametric quadrilateral element is used for both top and bottom layers. The thin weak layer is modeled using 1-D shear spring elements connecting the corresponding nodes on top and bottom layers.

The domain Ω is discretized into finite elements in the usual way. The elements are defined by the vector coordinates X_n of each node n. The unknown velocity field v is discretized in terms of the interpolation functions

$v = \sum_n v^n N_n$ where v_n is the velocity at node n, with the components V_i^n, and v is the vector of all the nodal coordinates, N_n is the global interpolation function which takes the value 1 at node n and 0 for any other node. The strain rate tensor can be evaluated

$$\dot{\varepsilon}_{ij} = \frac{1}{2}\sum_n [v_i^n \frac{\partial N_n(X)}{\partial X_j} + v_j^n \frac{\partial N_n(X)}{\partial X_i}] \quad (1)$$

The B operator is defined such that

$$\dot{\varepsilon}_{ij} = \sum_{nk} B_{ijnk} v_k^n \quad (2)$$

Therefore, we have $B'_{ijnk} = B_{ijnk} - \frac{1}{3} I_{ij} \sum_i B_{iink}$

(3)

Potential Energy of the system is given by

$$\Pi = \frac{1}{2}\int_\Omega \sigma^T \dot{\varepsilon}\, d\Omega - \int_\Omega v^T f d\Omega - \int_\Omega T^n v ds \quad (4)$$

Where Ω is the region occupied, $d\Omega$ is the boundry of Ω on which tractions are prescribed. Since, any external force other than gravity is not being considered, the last term can be neglected in further development. f is the body force, which can be resolved into two components, one parallel to the slope and other perpendicular to the slope.

VELOCITY INTERFACE CONDITIONS

The weak layer is modeled by 1-D shear spring elements. The reason for this is apparent. For continuity, the components of the velocity V_2^{top} and V_2^{bot} (where V_2^{top} and V_2^{bot} represents 2nd DOF of top and bottom node of 1-D element) must be equal. To achieve this interface condition, a penalty formulation is used. The stiffness(C) of the spring is taken very large in z-direction. It is approximately equal to 10^6 times of the largest diagonal element in the stiffness matrix .

The strain energy of the spring is equal to

$$U_s = \frac{1}{2} C(v_2^{top} - v_2^{bot})^2 \quad (5)$$

This strain energy contributes to the total potential energy. As a result, the modified potential energy of the system is

$$\Pi_m = \frac{1}{2}\int_\Omega \sigma^T \dot{\varepsilon}\, d\Omega - \int_\Omega v^T f\, d\Omega + \frac{1}{2}\sum_e C(v_2^{top} - v_2^{bot})^2 \quad (6)$$

SOLUTION PROCEDURE

The solution to the problem is obtained by minimization of the potential energy function. Since the constitutive law is nonlinear, equation for potential energy is to be linearized

$$R_n^{(v)} = \frac{\partial \Pi_m}{\partial v_n} = 0 \quad (7)$$

Newton-Raphson iterative method was employed for solving the system of nonlinear equations. Starting from an initial velocity $V^{(0)}$, at iteration r

$$v^{(r+1)} = v^{(r)} + \Delta r^{(r)} \quad (8)$$

Substituting this in eq. (7)

$$R(v^{(r+1)}) = R(v^{(r)} + \frac{\partial R}{\partial v}(v^{(r)})\Delta v^{(r)} \quad (9)$$

A line search procedure is necessary to speed up convergence.

$v^{(r+1)} = v^{(r)} - \theta[H^{(r)}]^{-1}R(v^{(r)})$ where θ is selected between 0 and 1 such that

$$R(v^{(r+1)}) \le R(v^{(r)})$$

VELOCITY CONVERGENCE

Convergence will occur if the norm of the residual velocities becomes less than TOLER times the norm of the total velocities. The parameter TOLER is chosen to be 0.01. The same convergence criterion was applied to unknown forces also. The norm of the residual velocities is calculated as

$\|\Delta v\| = \sqrt{\sum_i^p \Delta v_i^2}$ and the norm of total velocities $\|v\| = \sqrt{\sum_i^p v_i^2}$. Here, p is the total number of unknowns. Similar formula is applied for calculating the norm of residual forces and the norm of the total forces. The force vector is obtained by $\frac{\partial \Pi_m}{\partial v_k^n}$.

$$R_k^n = \frac{\partial \Pi_m}{\partial v_k^n} = \frac{\partial}{\partial v_k^n}(\sigma_{ij}\dot{\varepsilon}_{ij})$$

$$\sigma_{ij} = \frac{\partial \phi}{\partial \dot{\varepsilon}} = \frac{\partial \phi}{\partial \bar{\varepsilon}}\frac{\partial \bar{\varepsilon}}{\partial \varepsilon_{ij}} \quad (10)$$

First term of eq. (6) becomes

$$\frac{\partial \phi}{\partial \bar{\varepsilon}}\frac{1}{3\rho_r^2 \bar{\varepsilon}}[\frac{2}{c}\dot{e}_{pq} B'_{pqi} + \frac{1}{3f}\dot{\varepsilon}_{kk} B_{ppi}] \quad (11)$$

The Hessian is defined by $H = \frac{\partial R}{\partial V}$

$$\Rightarrow H_{Kk'} = \frac{\partial}{\partial v'_k}\int \frac{1}{3\rho_r^2}\frac{1}{\bar{\varepsilon}}\frac{\partial \phi}{\partial \bar{\varepsilon}} E_k dV \quad ;$$

$$E_k = [\frac{2}{c}\dot{e}_{rs} B'_{pqi} + \frac{1}{3f}\dot{\varepsilon}_{kk} B_{ppi}] \quad (12)$$

where k and k' goes from 1 to 2N. So $[H_{kk'}]_{2n\times 2n}$ for a single 'N' noded element is

$$H_{Kk'} = \frac{1}{3\rho_r^2}\frac{\partial}{\partial v'_k}(\frac{1}{\bar{\varepsilon}}\frac{\partial \phi}{\partial \bar{\varepsilon}})E_k + \frac{1}{3\rho_r^2}\frac{1}{\bar{\varepsilon}}\frac{\partial \phi}{\partial \bar{\varepsilon}}\frac{\partial E_k}{\partial v'_k} \quad (13)$$

First and second terms of Hessian are being represented by suffix a and b

$$[H_{kk'}]_a = \frac{1}{3\rho_r^2 \bar{\varepsilon}}\left[\frac{1}{3\rho_r^2 \bar{\varepsilon}^{-2}} - \frac{1}{\bar{\varepsilon}^{-2}}\frac{\partial \phi}{\partial \bar{\varepsilon}}\right] E_{uk'} E_{uk'}$$

$$[H_{kk'}]_b = \frac{1}{3\rho_r^2 \bar{\varepsilon}}\left[\frac{2}{c} B'_{rsk} B'_{rsk} + \frac{1}{3f} B'_{ssk} B_{ppk}\right] \tag{14}$$

Now considering the remaining terms in eq. (6), the force vectors can be obtained as

$$\frac{\partial}{\partial v_k}\left(\int v^T f d\Omega\right) = \int_\Omega N_k \rho_g \begin{Bmatrix} \sin\theta \\ \cos\theta \end{Bmatrix} d\Omega$$

$$\frac{\partial}{\partial v_k}\left[\sum_e \frac{1}{2} C (v_2^{top} - v_2^{bot})^2 = \sum C (v_2^{top} - v_2^{bot})(\delta^{top} - \delta^{bot}) \right. \tag{15}$$

where k is even.

RESULTS

The values of c, f and n has been computed with the help of 21 unconfined compression creep tests and 10 confined creep tests performed in a cold room. Temperature was kept at -10°C. Samples of snow were prepared and subjected to stresses in the range of 0.015 MPa to 0.06 MPa. Strain rate in the secondary creep region has been recorded at various time intervals. Due to the inherent property of snow constant strain-rate is not witnessed. Finding secondary strain-rate by averaging doesn't seem to be a errorfree method however, error may not be very high as the variation in strain-rate was low except on two occasions. Values of c, f and n were determined by least square method. The equation used were taken from Mishra et al (1996). Values of c, f and n were 1.35, 0.57 and 1.37 respectively.

Fig 1. Shear stress as a function of thickness of weak layer

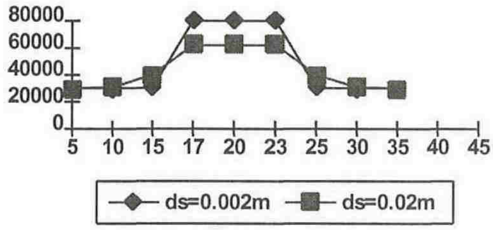


Fig 2. Deformation velocity as a function of length of weak layer

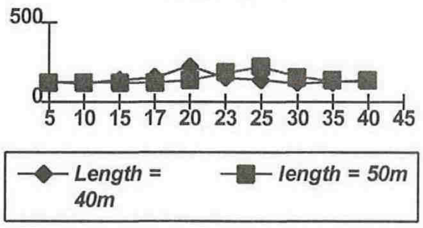


Fig 3. Deformation velocity as a function of thickness of weak layer

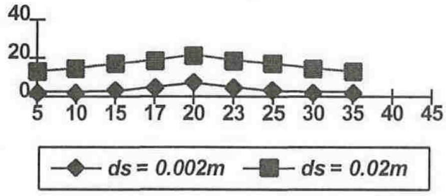
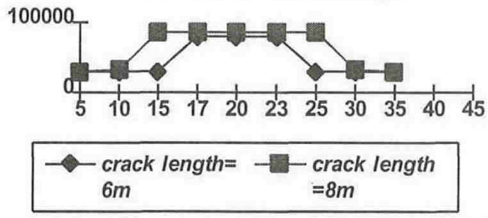


Fig 4. Shear stress as a function of crack length



The effect of the thickness of weak layer, length of the crack and span length of weak thin layer on shear stresses and deformation velocities were studied. Deformation velocity peak increases with increase of thickness of weak layer (Fig. 3) while the shear stress peak decreases with increase in thickness of weak layer (Fig 1). It can be observed that there is small increase in the peak deformation velocities and shear stress peak with increase in crack length (Fig. 4). There is no considerable difference in the location of deformation velocity and shear stress peak. When the length of the thin weak layer increases the peak deformation velocity and peak shear stress values shift towards right (Fig. 2).

CONCLUSION

It has been observed that shear stress is almost independent of thickness of weak layer. However, the thickness of weak layer plays an important role in crack propagation. The strain-rates corresponding to the thickness of weak layer more than 0.005 m is always less than the critical strain-rate for crack propagation. It is unlikely that crack will ever propagate if thickness is more than 0.005m. To know the critical crack length to start propagation, thickness of the weak layer was kept constant at 2 mm. The critical strain rates for ductile and brittle fractures are 10^{-4} S^{-1} and 10^{-3} S^{-1} respectively (Narita). The critical crack length to start propagation is found to be a_{cr} = 6.32 m. The study shows larger crack length and smaller thickness lead to higher strain-rates which in turn means shorter time to fracture.

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